Groups in which every Subgroup is either Abelian or Dihedral.

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As the groups in which every subgroup is abelian are known* we shall assume that every group G under consideration contains at least one non-abelian dihedral subgroup. Our first object is to prove that G is always solvable. Since every subgroup and every quotient group of G is either abelian, or has the properties which have been assumed with respect to G, it follows that G is either solvable or there is some G which is simple. It is therefore only necessary to show that G cannot be a simple group. We shall do this by assuming that G is simple and showing that this assumption necessarily leads to a contradiction. Until G is proved solvable it will be assumed that it is simple.

Suppose that G is represented as a substitution group in such a way that a Sylow subgroup of odd order (P) is composed of all the substitutions of G which do not involve a given letter. The subgroup P is abelian. If it were maximal G would be of degree n and class $n-1$, since P would be of degree $n-1$ and could have only the identity in common with any of its conjugates. As a group of degree n and class $n-1$ contains an invariant subgroup, P cannot be maximal and hence it must be contained in a dihedral subgroup of G. The latter statement follows from the fact that no abelian subgroup can be maximal in a simple group. Hence we have proved the lemma that *every Sylow subgroup of odd order contained in G is cyclic and every abelian subgroup of G is contained in some dihedral subgroup whenever G is simple.*

Since the Sylow subgroup of order 2^m is contained in a dihedral group it must itself be dihedral and hence involves operators of order 2^{m-1}. Suppose now that G is represented as a primitive substitution group in such a way that the subgroup (G_1) which is composed of all its substitutions which do not involve

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a given letter involves a Sylow subgroup of order 2^m. There is one invariant operator (s_1) of order 2 in G_1* and all its operators whose order exceeds 2 are of degree $n-1$, n being the degree of G. We proceed to prove that the degree of s_1 is less than $n-1$.

Since G is simple and G_1 is a maximal subgroup, s_1 is not commutative with any operators of G, except those of G_1. Hence s_1 has n conjugates under G. These n conjugates include all the operators of G which are powers of operators of order 2^α, $\alpha > 1$, since all the cyclic subgroups of order 2^α are conjugate under G. If G contained any operators of order 2 besides those which are conjugate with s_1 it would contain negative substitutions if it were represented as a transitive substitution group in such a way that one of these operators of order 2 would generate the subgroup involving all the substitutions which do not contain a given letter. That is, G contains a single set of conjugates of order 2 and the number of these conjugates is n.

From the preceding paragraph it follows that G would be of class $n-1$ if s_1 were of degree $n-1$. As a simple group cannot be of degree n and of class $n-1$ it follows that the degree of s_1 is less than $n-1$. The number of four-groups in G_1 is $g_1/4$, g_1 being the order of G_1. As every operator of order 2 in G is contained in exactly $g_1/4$ four-groups, the total number of these subgroups is $ng_1/12 = g/12$, g being the order of G. If they would all be conjugate each of them would be invariant under a group of order 12. As such a subgroup is neither dihedral nor contained in a dihedral group it follows that G cannot be simple unless its order is divisible by 8.

If g were divisible by 8 G would contain two operators of order 4, (t_1, t_2) such that the square of each would transform the other into its inverse. That is, we would have

$$t_1^4 = t_2^4 = 1, \qquad t_1^2 t_2 t_1^2 = t_2^3, \qquad t_2^2 t_1 t_2^2 = t_1^3, \qquad t_1^2 \neq 1.$$

From these relations it follows that $\{t_1^2, t_2^2\}$ is the four-group. Moreover the equations

$$(t_1 t_2)^{-1} t_1^2 t_1 t_2 = t_1^2 t_2^2, \qquad (t_1 t_2)^{-1} t_1^2 t_2^2 t_1 t_2 = t_2^2,$$

show that the subgroup generated by $s_1 s_2$, s_1^2, s_2^2 is neither abelian nor dihedral. This proves that G is composite. In other words, *If every non-abelian sub-group of a group is dihedral the group is always solvable.*

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* The order of G_1 is divisible by 4 since the order of G cannot be twice an odd number.

If the order of a dihedral group is divisible by 4, its non-invariant operators of order 2 may be replaced by operators of order 4 which transform the operators of the group in the same way as the operators of order 2 did. That is, the group of cogredient isomorphisms remains unchanged. The octic and the quaternion groups are simple illustrations of such groups. If we call the latter type associate-dihedral, the arguments used above prove also that *if every non-abelian sub-group of a group is either dihedral or associate-dihedral the group is solvable.*

Since G is solvable it contains an invariant subgroup of prime index (p). If $p > 2$ the corresponding invariant subgroup (H) involves all the operators of G whose order is of the form 2^{α}. There is some operator (s) of order p^{β} which is found in G but not in H. If H were abelian it would contain some operators of order 2 which would transform an operator of order p into its inverse since G contains at least one non-abelian dihedral subgroup. This is impossible since every two operators of order 2 in H are supposed to be commutative. Hence H must be dihedral whenever $p > 2$. As s is commutative with every operator in the cyclic subgroup of half the order of H it follows that *G is either the direct product of the octic group and a cyclic group of order p, or it is the direct product of a cyclic group of order p and a dihedral group of order $2q$, q being an odd prime, whenever $p > 2$.*

When $p = 2$ G may be a dihedral group. As the properties of such a G are well known we shall assume for the present that G is not dihedral but that H is abelian. Some operators of order 2 which are contained in the non-abelian dihedral subgroup (D) cannot occur in H since H is abelian. We may therefore assume that S is of order 2. The considerations of this case may be simplified by means of the following theorem, which is of interest on account of its great generality. *If an operator of order 2 transforms a group of order 2^m into itself and is commutative with only two of its operators the group of order 2^m is cyclic, dihedral, or associate dihedral.*

The proof of this theorem follows almost directly from the facts that the given group contains (whenever $m > 3$) an abelian non-cyclic invariant subgroup of order 8 whenever it contains more than one cyclic subgroup of order 8,* and an operator of order 2 which transforms an abelian non-cyclic group of order 8 into itself is commutative with at least four of its operators. The group of order 2^m, $m > 3$, which contains only one cyclic group of order 8 and is not

*Transactions of the American Mathematical Society, vol. 6 (1905), p. 60.

mentioned in the theorem can clearly not be transformed into itself by an operator of order 2 which is commutative with only two of its operators. Hence the proof of the theorem is complete since the four-group is dihedral and the octic and quaternion groups are respectively dihedral and associate dihedral.

If G would contain an invariant operator of odd prime order it would also involve an invariant subgroup of odd prime index since the given invariant operator could not be contained in D. As this case has been considered we may assume that s transforms into its inverse every operator of odd prime order contained in H. The truth of the last statement follows directly from the theorem: If an operator (t) transforms commutative operators among themselves it is commutative with the continued product of any complete set of conjugates under t. In particular, *if an operator of order two transforms any operator into one which is commutative with it, the operator of order two is itself commutative with the product of this operator and its conjugate.*

Suppose that G contains more than two invariant operators. From the preceding paragraph it follows that it is only necessary to consider the case when there is an invariant operator (i) of order 2^β in G and i does not occur in D. Since the direct product of the group of order 2 and a dihedral group whose order is not divisible by 4 is dihedral, we may assume that the order of D is divisible by 4. If i is of order 2 all the subgroups of D whose orders are divisible by 4 must be abelian. Hence the order of D is not divisible by 8 unless D is the octic group. As the order of D is supposed to be divisible by 4 it can contain only one subgroup of odd order. That is, *if G contains at least two invariant operators of order 2 it is the direct product of a group of order 2 and either the octic group or the dihedral group of order $4p$, p being an odd prime.*

If the order of i were four and i^2 were not in D we would have the case considered above and hence such a G is impossible. The assumption that i^2 is in D also leads to a contradiction since the product of i and a non-invariant operator of order 2 in D would be of order 4, and this product together with the cyclic subgroup of half the order of D would generate an associate dihedral group. That is the order of i can not exceed 2. It remains therefore to consider the G's which involve at most two invariant operators including the identity.

By means of the preceding theorems it is not difficult to complete the consideration of all these groups when H is abelian. Since G is not dihedral and does not contain more than two invariant operators the order of H cannot be

of the form 2^{α}. As the Sylow subgroup of order 2^m, $m > 2$, in G would be dihedral it follows that G would be dihedral. That is, it is impossible to construct a non-dihedral G when H is abelian and G does not involve more than two invariant operators unless G is of odd order. In this case G may be the abelian group of order p^2 and of type (1, 1), p being an odd prime.

When H is dihedral and its cyclic subgroup (C) is of order $2^{\alpha 0} p_1^{\alpha 1} p_2^{\alpha 2} \ldots p_\lambda^{\alpha\lambda}$ we may choose s in such a way that all the operators of G which are not in H are of order 4, provided $\lambda = 1$, $\alpha_0 = 0$, $\alpha_1 = 1$, and $p \equiv 1 \bmod 4$. This is the only possible type when s^2 is not invariant under C. If s were commutative with more than two operators of C without being commutative with all of them, the subgroup generated by s and C would not be dihedral. Hence s is either commutative with all the operators of C, or it transforms each one of them into its inverse. In the former case we could replace s by the product obtained by multiplying s into some non-invariant operator of order 2 in H. As the latter can be so selected that the order of this product is again of the form 2^{β}, it may be assumed that s is of order 2 and transforms each operator of C into its inverse.

If s were commutative with a non-invariant operator of order 2 in H the product of s into this operator would be invariant under G. This case was considered above. If s were not commutative with such an operator it would have to transform the non-invariant operators of order 2 in H into themselves multiplied by operators whose order is divisible by $2^{\alpha 0}$, $\alpha_0 > 0$. In this case G would involve operators of order $2^{\alpha 0+1} p_1^{\alpha 1} p_2^{\alpha 2} \ldots p_\lambda^{\alpha\lambda}$ and hence would be dihedral. Hence H cannot be dihedral when s^2 is found in C. We have now considered all the possible cases and may express the results as follows: Every non-abelian subgroup of a dihedral group is dihedral. If G is both non-abelian and non-dihedral, and if every non-abelian subgroup of G is dihedral then G belongs to one of the following five types of groups:

1) The direct product of the octic group and a group of order p, p being any prime number,
2) The direct product of the dihedral group of order $2q$, q being an odd prime, and the group of order p,
3) The direct product of the dihedral group of order $4q$ and the group of order 2,

4) The group obtained by extending the abelian group of order p^2, $p>2$, and of type (1, 1) by means of an operator of order 2 which transforms each operator of this abelian group into its inverse.
5) The group of order $4q$, q being prime and $\equiv 1 \bmod 4$, which is contained in the holomorph of the group of order q.

The first three types contain invariant operators while there is no invariant operator besides the identity in either of the last two types. None of these groups contains invariant operators of order p^2, p being any prime. If any other group contains a non-abelian dihedral subgroup it is either dihedral or it involves non-abelian subgroups which are not dihedral.